

# Generalised matricvariate $T$ -distribution

**José A. Díaz-García \***

Department of Statistics and Computation  
Universidad Autonoma Agraria Antonio Narro  
25350 Buenavista, Saltillo, Coahuila, Mexico  
E-mail: jadiaz@uaaan.mx

**Ramón Gutiérrez-Sánchez**

Department of Statistics and O.R  
University of Granada  
Granada 18071, Spain  
E-mail: ramongs@ugr.es

## Abstract

Assuming Kotz-Riesz type I and II distributions and their corresponding independent Riesz distributions the associated generalised matricvariate  $T$  distributions, termed matricvariate  $T$ -Riesz distributions for real normed division algebras are obtained with respect to the Lebesgue measure. In addition some of their properties are also studied.

## 1 Introduction

Since the early 80's years the elliptical distribution family has been used as alternative sampling distribution to assumption of normality. The elliptical distribution family, within other qualities of interest, are very attractive because if it is assumed that the random matrix  $\mathbf{X}$  has a matrix multivariate elliptical distribution, then the distributions of their many matrix functions,  $\mathbf{Y} = f(\mathbf{X})$ , are invariant under the family of elliptical distributions, furthermore, such distributions coincide with those obtained when  $\mathbf{X}$  is distributed according to a matrix multivariate normal distribution, see Fang and Zhang (1990) and Gupta and Varga (1993).

However it should be noted that the invariance described above occurs when it is assumed probabilistically dependent. This is, if the matrix  $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$  has a matrix multivariate elliptical distribution, observing that  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are probabilistically dependent (note that  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are probabilistically independent if  $\mathbf{X}$  has a matrix multivariate normal distribution, Gupta and Varga (1993)). Then if  $\mathbf{X}'$  denotes the transpose of  $\mathbf{X}$  and if  $\mathbf{A}$  is non-negative definite matrix and  $\mathbf{A}^{1/2}$  denotes its non-negative definite square root, such that  $\mathbf{A} = \mathbf{A}^{1/2} \mathbf{A}^{1/2}$  (Muirhead, 1982), then:

---

\*Corresponding author

**Key words.** Matricvariate;  $T$ -distribution; Riesz distribution; Kotz-Riesz distribution; real, complex, quaternion and octonion random matrices; real normed division algebras.  
2000 Mathematical Subject Classification. 15A23; 15B33; 15A09; 15B52; 60E05

- Let  $\mathbf{T} = \mathbf{X}_1(\mathbf{X}_2'\mathbf{X}_2)^{-1/2}$ . It is said that  $\mathbf{T}$  is distributed according to a matricvariate<sup>1</sup>  $T$ -distribution.
- Let  $\mathbf{F} = (\mathbf{X}_2'\mathbf{X}_2)^{-1/2}(\mathbf{X}_1'\mathbf{X}_1)(\mathbf{X}_2'\mathbf{X}_2)^{-1/2}$ . It is said that  $\mathbf{F}$  has a matricvariate beta type II distribution, and
- let  $\mathbf{B} = (\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2)^{-1/2}(\mathbf{X}_1'\mathbf{X}_1)(\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2)^{-1/2}$ . It is said that  $\mathbf{B}$  is distributed according to a matricvariate beta type I distribution,

where the matricvariate  $T$ , beta type II and beta type I distributions are the same distributions as those obtained when  $\mathbf{X}$  has a matrix multivariate normal distribution, see Fang and Zhang (1990) and Gupta and Varga (1993).

Unfortunately, in some situations,  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are assumed independent. This situation can occur in the context of multivariate Bayesian inference, Press (1982). For example, suppose that a particular distribution depend of two matrix parameters, say  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_2$ ; and is assumed that  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_2$  have a distribution as the marginal distributions of  $\mathbf{X}_1$  and  $\mathbf{X}_2$  respectively, but in this case is assumed that  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_2$  are *independent*. Under this hypothesis, one is interested in finding the priori distribution of a parameter type  $\mathbf{T}$ ,  $\mathbf{F}$  or  $\mathbf{B}$  in terms of the priori distribution of  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_2$ . In this case priori distributions of  $\mathbf{T}$ ,  $\mathbf{F}$  or  $\mathbf{B}$  are different from those obtained under dependence, moreover, such priori distributions are different under each particular elliptical distribution assumed.

Of particular interest is the matrix multivariate elliptical distribution termed Kotz-Riesz distribution by its relation with the Riesz distribution, Díaz-García (2013c). If  $\mathbf{X}$  is distributed according to matrix multivariate Kotz-Riesz, then the matrix  $\mathbf{V} = \mathbf{X}'\mathbf{X}$  has a Riesz distribution. The Riesz distributions, were first introduced by Hassairi and Lajmi (2001) under the name of Riesz natural exponential family (Riesz NEF); it was based on a special case of the so-called Riesz measure from Faraut and Korányi (1994, p.137). This Riesz distribution generalises the matrix multivariate gamma and Wishart distributions, containing them as particular cases.

In analogy with the case of  $T$ -distribution obtained under normality, there exist two possible generalisations of it when is assumed a Kotz-Riesz distribution, the matricvariate  $T$ -distribution and the matrix multivariate  $T$ -distribution, see Díaz-García and Gutiérrez-Jáimez (2012). In this paper it is addressed under the case of the distribution termed, matricvariate  $T$ -Riesz distribution.

Resuming, some basic concepts and the notation of abstract algebra and Jacobians are summarised in Section 2. The nonsingular central matricvariate  $T$ -Riesz and the corresponding generalised beta type II distributions and some of their basic properties are studied in Section 3. Finally, the joint densities of the singular values are derived in Section 4. We would like emphasize that all these results are derived for real normed division algebras.

## 2 Preliminary results

A detailed discussion of real normed division algebras may be found in Baez (2002) and Neukirch *et al.* (1990). For your convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes: Let  $\mathbb{F}$  be a field. An *algebra*  $\mathfrak{A}$  over  $\mathbb{F}$  is a pair  $(\mathfrak{A}; m)$ , where  $\mathfrak{A}$  is a *finite-dimensional vector space* over  $\mathbb{F}$  and *multiplication*  $m : \mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$  is an  $\mathbb{F}$ -bilinear

---

<sup>1</sup>The term matricvariate distribution was first introduced Dickey (1967), but the expression matrix-variate distribution or matrix variate distribution or matrix multivariate distribution was later used to describe any distribution of a random matrix, see Gupta and Nagar (2000), and references therein. When the density function of a random matrix is written only in terms of determinant operator and  $q_\kappa(\cdot)$  (defined in the next section) then the matricvariate designation shall be used

map; that is, for all  $\lambda \in \mathbb{F}$ ,  $x, y, z \in \mathfrak{A}$ ,

$$\begin{aligned} m(x, \lambda y + z) &= \lambda m(x, y) + m(x, z) \\ m(\lambda x + y, z) &= \lambda m(x, z) + m(y, z). \end{aligned}$$

Two algebras  $(\mathfrak{A}; m)$  and  $(\mathfrak{E}; n)$  over  $\mathbb{F}$  are said to be *isomorphic* if there is an invertible map  $\phi : \mathfrak{A} \rightarrow \mathfrak{E}$  such that for all  $x, y \in \mathfrak{A}$ ,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write  $m(x, y) = xy$  for all  $x, y \in \mathfrak{A}$ .

Let  $\mathfrak{A}$  be an algebra over  $\mathbb{F}$ . Then  $\mathfrak{A}$  is said to be

1. *alternative* if  $x(xy) = (xx)y$  and  $x(yy) = (xy)y$  for all  $x, y \in \mathfrak{A}$ ,
2. *associative* if  $x(yz) = (xy)z$  for all  $x, y, z \in \mathfrak{A}$ ,
3. *commutative* if  $xy = yx$  for all  $x, y \in \mathfrak{A}$ , and
4. *unital* if there is a  $1 \in \mathfrak{A}$  such that  $x1 = x = 1x$  for all  $x \in \mathfrak{A}$ .

If  $\mathfrak{A}$  is unital, then the identity 1 is uniquely determined.

An algebra  $\mathfrak{A}$  over  $\mathbb{F}$  is said to be a *division algebra* if  $\mathfrak{A}$  is nonzero and  $xy = 0_{\mathfrak{A}} \Rightarrow x = 0_{\mathfrak{A}}$  or  $y = 0_{\mathfrak{A}}$  for all  $x, y \in \mathfrak{A}$ .

The term “division algebra”, comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let  $\mathfrak{A}$  be an algebra over  $\mathbb{F}$ . Then  $\mathfrak{A}$  is a division algebra if, and only if,  $\mathfrak{A}$  is nonzero and for all  $a, b \in \mathfrak{A}$ , with  $b \neq 0_{\mathfrak{A}}$ , the equations  $bx = a$  and  $yb = a$  have unique solutions  $x, y \in \mathfrak{A}$ .

In the sequel we assume  $\mathbb{F} = \mathbb{R}$  and consider classes of division algebras over  $\mathbb{R}$  or “*real division algebras*” for short.

We introduce the algebras of *real numbers*  $\mathbb{R}$ , *complex numbers*  $\mathbb{C}$ , *quaternions*  $\mathbb{H}$  and *octonions*  $\mathbb{O}$ . Then, if  $\mathfrak{A}$  is an alternative real division algebra, then  $\mathfrak{A}$  is isomorphic to  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  or  $\mathbb{O}$ .

Let  $\mathfrak{A}$  be a real division algebra with identity 1. Then  $\mathfrak{A}$  is said to be *normed* if there is an inner product  $(\cdot, \cdot)$  on  $\mathfrak{A}$  such that

$$(xy, xy) = (x, x)(y, y) \quad \text{for all } x, y \in \mathfrak{A}.$$

If  $\mathfrak{A}$  is a *real normed division algebra*, then  $\mathfrak{A}$  is isomorphic to  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  or  $\mathbb{O}$ .

There are exactly four normed division algebras: real numbers ( $\mathbb{R}$ ), complex numbers ( $\mathbb{C}$ ), quaternions ( $\mathbb{H}$ ) and octonions ( $\mathbb{O}$ ), see Baez (2002). We take into account that,  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  and  $\mathbb{O}$  are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let  $\mathfrak{A}$  be a division algebra over the real numbers. Then  $\mathfrak{A}$  has dimension either 1, 2, 4 or 8. Other branches of mathematics used the parameters  $\alpha = 2/\beta$  and  $t = \beta/4$ , see Edelman and Rao (2005) and Kabe (1984), respectively.

Finally, observe that

- $\mathbb{R}$  is a real commutative associative normed division algebras,
- $\mathbb{C}$  is a commutative associative normed division algebras,
- $\mathbb{H}$  is an associative normed division algebras,
- $\mathbb{O}$  is an alternative normed division algebras.

Let  $\mathfrak{L}_{m,n}^{\beta}$  be the set of all  $m \times n$  matrices of rank  $m \leq n$  over  $\mathfrak{A}$  with  $m$  distinct positive singular values, where  $\mathfrak{A}$  denotes a *real finite-dimensional normed division algebra*. Let

$\mathfrak{A}^{m \times n}$  be the set of all  $m \times n$  matrices over  $\mathfrak{A}$ . The dimension of  $\mathfrak{A}^{m \times n}$  over  $\mathfrak{R}$  is  $\beta mn$ . Let  $\mathbf{A} \in \mathfrak{A}^{m \times n}$ , then  $\mathbf{A}^* = \mathbf{A}^T$  denotes the usual conjugate transpose.

Table 1 sets out the equivalence between the same concepts in the four normed division algebras.

| Table 1: Notation |              |                      |                       |                           |
|-------------------|--------------|----------------------|-----------------------|---------------------------|
| Real              | Complex      | Quaternion           | Octonion              | Generic notation          |
| Semi-orthogonal   | Semi-unitary | Semi-symplectic      | Semi-exceptional type | $\mathcal{V}_{m,n}^\beta$ |
| Orthogonal        | Unitary      | Symplectic           | Exceptional type      | $\mathfrak{U}^\beta(m)$   |
| Symmetric         | Hermitian    | Quaternion hermitian | Octonion hermitian    | $\mathfrak{S}_m^\beta$    |

It is denoted by  $\mathfrak{S}_m^\beta$  the real vector space of all  $\mathbf{S} \in \mathfrak{A}^{m \times m}$  such that  $\mathbf{S} = \mathbf{S}^*$ . In addition, let  $\mathfrak{P}_m^\beta$  be the *cone of positive definite matrices*  $\mathbf{S} \in \mathfrak{A}^{m \times m}$ . Thus,  $\mathfrak{P}_m^\beta$  consist of all matrices  $\mathbf{S} = \mathbf{X}\mathbf{X}^*$ , with  $\mathbf{X} \in \mathfrak{L}_{m,n}^\beta$ ; then  $\mathfrak{P}_m^\beta$  is an open subset of  $\mathfrak{S}_m^\beta$ .

Let  $\mathfrak{D}_m^\beta$  be the *diagonal subgroup* of  $\mathcal{L}_{m,m}^\beta$  consisting of all  $\mathbf{D} \in \mathfrak{A}^{m \times m}$ ,  $\mathbf{D} = \text{diag}(d_1, \dots, d_m)$ . Let  $\mathfrak{T}_U^\beta(m)$  be the subgroup of all *upper triangular* matrices  $\mathbf{T} \in \mathfrak{A}^{m \times m}$  such that  $t_{ij} = 0$  for  $1 < i < j \leq m$ .

The set of matrices  $\mathbf{H}_1 \in \mathfrak{F}^{m \times n}$  such that  $\mathbf{H}_1 \mathbf{H}_1^* = \mathbf{I}_m$  is a manifold denoted  $\mathcal{V}_{m,n}^\beta$ , is termed the *Stiefel manifold* ( $\mathbf{H}_1$  is also known as *semi-orthogonal*).

For any matrix  $\mathbf{X} \in \mathfrak{A}^{m \times n}$ ,  $d\mathbf{X}$  denotes the *matrix of differentials*  $(dx_{ij})$ . Finally, we define the *measure* or volume element  $(d\mathbf{X})$  when  $\mathbf{X} \in \mathfrak{A}^{m \times n}$ ,  $\mathfrak{S}_m^\beta$ ,  $\mathfrak{D}_m^\beta$  or  $\mathcal{V}_{m,n}^\beta$ , see Díaz-García and Gutiérrez-Jáimez (2011).

If  $\mathbf{X} \in \mathfrak{A}^{m \times n}$  then  $(d\mathbf{X})$  (the Lebesgue measure in  $\mathfrak{A}^{m \times n}$ ) denotes the exterior product of the  $\beta mn$  functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^m \bigwedge_{j=1}^n dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^\beta dx_{ij}^{(k)}.$$

If  $\mathbf{S} \in \mathfrak{S}_m^\beta$  (or  $\mathbf{S} \in \mathfrak{T}_U^\beta(m)$  with  $t_{ii} > 0$ ,  $i = 1, \dots, m$ ) then  $(d\mathbf{S})$  (the Lebesgue measure in  $\mathfrak{S}_m^\beta$  or in  $\mathfrak{T}_U^\beta(m)$ ) denotes the exterior product of the  $m(m-1)\beta/2 + m$  functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^m ds_{ii} \bigwedge_{i < j}^m \bigwedge_{k=1}^\beta ds_{ij}^{(k)}.$$

Observe, that for the Lebesgue measure  $(d\mathbf{S})$  defined thus, it is required that  $\mathbf{S} \in \mathfrak{P}_m^\beta$ , that is,  $\mathbf{S}$  must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If  $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$  then  $(d\mathbf{\Lambda})$  (the Lebesgue measure in  $\mathfrak{D}_m^\beta$ ) denotes the exterior product of the  $\beta m$  functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^n \bigwedge_{k=1}^\beta d\lambda_i^{(k)}.$$

If  $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$  is such that  $\mathbf{H}_1 = (\mathbf{h}_1^*, \dots, \mathbf{h}_m^*)^*$ , where  $\mathbf{h}_i$ ,  $i = 1, \dots, m$  are their rows, then

$$(\mathbf{H}_1 d\mathbf{H}_1^*) = \bigwedge_{i=1}^m \bigwedge_{j=i+1}^n \mathbf{h}_j^* d\mathbf{h}_i, \quad (1)$$

where the partitioned matrix  $\mathbf{H} = (\mathbf{H}_1^* | \mathbf{H}_2^*)^* = (\mathbf{h}_1^*, \dots, \mathbf{h}_m^* | \mathbf{h}_{m+1}^*, \dots, \mathbf{h}_n^*)^* \in \mathfrak{U}^\beta(n)$ , with  $\mathbf{H}_2 = (\mathbf{h}_{m+1}, \dots, \mathbf{h}_n)$ . It can be proved that this differential form does not depend on the

choice of the  $\mathbf{H}_2$  matrix. When  $n = 1$ ;  $\mathcal{V}_{m,1}^\beta$  defines the unit sphere in  $\mathfrak{A}^m$ . This is, of course, an  $(m-1)\beta$ - dimensional surface in  $\mathfrak{A}^m$ .

The surface area or volume of the Stiefel manifold  $\mathcal{V}_{m,n}^\beta$  is

$$\text{Vol}(\mathcal{V}_{m,n}^\beta) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta} (\mathbf{H}_1 d\mathbf{H}_1^*) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^\beta[n\beta/2]}, \quad (2)$$

where  $\Gamma_m^\beta[a]$  denotes the multivariate *Gamma function* for the space  $\mathfrak{S}_m^\beta$ . This can be obtained as a particular case of the *generalised gamma function of weight  $\kappa$*  for the space  $\mathfrak{S}_m^\beta$  with  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ , taking  $\kappa = (0, 0, \dots, 0)$  and which for  $\text{Re}(a) \geq (m-1)\beta/2 - k_m$  is defined by, see Gross and Richards (1987),

$$\begin{aligned} \Gamma_m^\beta[a, \kappa] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}) (d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2] \\ &= [a]_\kappa^\beta \Gamma_m^\beta[a], \end{aligned} \quad (3)$$

where  $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$ ,  $|\cdot|$  denotes the determinant, and for  $\mathbf{A} \in \mathfrak{S}_m^\beta$

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}} \quad (5)$$

with  $\mathbf{A}_p = (a_{rs})$ ,  $r, s = 1, 2, \dots, p$ ,  $p = 1, 2, \dots, m$  is termed the *highest weight vector*, see Gross and Richards (1987). Also,

$$\begin{aligned} \Gamma_m^\beta[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\ &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - (i-1)\beta/2], \end{aligned}$$

and  $\text{Re}(a) > (m-1)\beta/2$ .

In other branches of mathematics the *highest weight vector*  $q_\kappa(\mathbf{A})$  is also termed the *generalised power* of  $\mathbf{A}$  and is denoted as  $\Delta_\kappa(\mathbf{A})$ , see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Additional properties of  $q_\kappa(\mathbf{A})$ , which are immediate consequences of the definition of  $q_\kappa(\mathbf{A})$  and the following property 1, are:

1. if  $\lambda_1, \dots, \lambda_m$ , are the eigenvalues of  $\mathbf{A}$ , then

$$q_\kappa(\mathbf{A}) = \prod_{i=1}^m \lambda_i^{k_i}. \quad (6)$$

- 2.

$$q_\kappa(\mathbf{A}^{-1}) = q_\kappa^{-1}(\mathbf{A}) = q_{-\kappa}(\mathbf{A}), \quad (7)$$

3. if  $\kappa = (p, \dots, p)$ , then

$$q_\kappa(\mathbf{A}) = |\mathbf{A}|^p, \quad (8)$$

in particular if  $p = 0$ , then  $q_\kappa(\mathbf{A}) = 1$ .

4. if  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ , then

$$q_{\kappa+\tau}(\mathbf{A}) = q_{\kappa}(\mathbf{A})q_{\tau}(\mathbf{A}), \quad (9)$$

in particular if  $\tau = (p, p, \dots, p)$ , then

$$q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_{\kappa}(\mathbf{A}). \quad (10)$$

5. Finally, for  $\mathbf{B} \in \mathfrak{A}^{m \times m}$  in such a manner that  $\mathbf{C} = \mathbf{B}^* \mathbf{B} \in \mathfrak{S}_m^{\beta}$ ,

$$q_{\kappa}(\mathbf{B} \mathbf{A} \mathbf{B}^*) = q_{\kappa}(\mathbf{C}) q_{\kappa}(\mathbf{A}) \quad (11)$$

and

$$q_{\kappa}(\mathbf{B}^{-1} \mathbf{A} \mathbf{B}^{*-1}) = (q_{\kappa}(\mathbf{C}))^{-1} q_{\kappa}(\mathbf{A}). \quad (12)$$

**Remark 2.1.** Let  $\mathcal{P}(\mathfrak{S}_m^{\beta})$  denote the algebra of all polynomial functions on  $\mathfrak{S}_m^{\beta}$ , and  $\mathcal{P}_k(\mathfrak{S}_m^{\beta})$  the subspace of homogeneous polynomials of degree  $k$  and let  $\mathcal{P}^{\kappa}(\mathfrak{S}_m^{\beta})$  be an irreducible subspace of  $\mathcal{P}(\mathfrak{S}_m^{\beta})$  such that

$$\mathcal{P}_k(\mathfrak{S}_m^{\beta}) = \sum_{\kappa} \bigoplus \mathcal{P}^{\kappa}(\mathfrak{S}_m^{\beta}).$$

Note that  $q_{\kappa}$  is a homogeneous polynomial of degree  $k$ , moreover  $q_{\kappa} \in \mathcal{P}^{\kappa}(\mathfrak{S}_m^{\beta})$ , see Gross and Richards (1987).  $\square$

In (4),  $[a]_{\kappa}^{\beta}$  denotes the generalised Pochhammer symbol of weight  $\kappa$ , defined as

$$\begin{aligned} [a]_{\kappa}^{\beta} &= \prod_{i=1}^m (a - (i-1)\beta/2)_{k_i} \\ &= \frac{\pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2]}{\Gamma_m^{\beta}[a]} \\ &= \frac{\Gamma_m^{\beta}[a, \kappa]}{\Gamma_m^{\beta}[a]}, \end{aligned}$$

where  $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$  and

$$(a)_i = a(a+1) \cdots (a+i-1),$$

is the standard Pochhammer symbol.

An alternative definition of the generalised gamma function of weight  $\kappa$  is proposed by Khatri (1966), which is defined as

$$\Gamma_m^{\beta}[a, -\kappa] = \int_{\mathbf{A} \in \mathfrak{P}_m^{\beta}} \operatorname{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_{\kappa}(\mathbf{A}^{-1}) (d\mathbf{A}) \quad (13)$$

$$\begin{aligned} &= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a - k_i - (m-i)\beta/2] \\ &= \frac{(-1)^k \Gamma_m^{\beta}[a]}{[-a + (m-1)\beta/2 + 1]_{\kappa}^{\beta}}, \end{aligned} \quad (14)$$

where  $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$ .

In addition consider the following generalised beta functions, see Faraut and Korányi (1994, p. 130) and Díaz-García (2013b),

$$\begin{aligned}
\mathcal{B}_m^\beta[a, \kappa; b, \tau] &= \int_{\mathbf{0} < \mathbf{S} < \mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S}) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{I}_m - \mathbf{S}) (d\mathbf{S}) \\
&= \int_{\mathbf{R} \in \mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{\kappa+\tau}^{-1}(\mathbf{I}_m + \mathbf{R}) (d\mathbf{R}) \\
&= \frac{\Gamma_m^\beta[a, \kappa] \Gamma_m^\beta[b, \tau]}{\Gamma_m^\beta[a+b, \kappa+\tau]}, \tag{15}
\end{aligned}$$

where  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers,  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers,  $\text{Re}(a) > (m-1)\beta/2 - k_m$  and  $\text{Re}(b) > (m-1)\beta/2 - t_m$ . Similarly,

$$\begin{aligned}
\mathcal{B}_m^\beta[a, -\kappa; b, -\tau] &= \int_{\mathbf{0} < \mathbf{S} < \mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa^{-1}(\mathbf{S}) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau^{-1}(\mathbf{I}_m - \mathbf{S}) (d\mathbf{S}) \\
&= \int_{\mathbf{R} \in \mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa^{-1}(\mathbf{R}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{\kappa+\tau}(\mathbf{I}_m + \mathbf{R}) (d\mathbf{R}) \\
&= \frac{\Gamma_m^\beta[a, -\kappa] \Gamma_m^\beta[b, -\tau]}{\Gamma_m^\beta[a+b, -\kappa-\tau]},
\end{aligned}$$

where  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers,  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers,  $\text{Re}(a) > (m-1)\beta/2 + k_1$  and  $\text{Re}(b) > (m-1)\beta/2 + t_1$ .

Now, we show four Jacobians in terms of the  $\beta$  parameter, which are proposed as extensions of real, complex or quaternion cases, see Díaz-García and Gutiérrez-Jáimez (2011).

**Proposition 2.1.** *Let  $\mathbf{X}$  and  $\mathbf{Y} \in \mathcal{L}_{m,n}^\beta$  be matrices of functionally independent variables, and let  $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{B} + \mathbf{C}$ , where  $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$ ,  $\mathbf{B} \in \mathcal{L}_{n,n}^\beta$  and  $\mathbf{C} \in \mathcal{L}_{m,n}^\beta$  are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^* \mathbf{A}|^{\beta n/2} |\mathbf{B}^* \mathbf{B}|^{\beta m/2} (d\mathbf{X}). \tag{16}$$

**Proposition 2.2.** *Let  $\mathbf{X}$  and  $\mathbf{Y} \in \mathfrak{S}_m^\beta$  be matrices of functionally independent variables, and let  $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{C}$ , where  $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$  and  $\mathbf{C} \in \mathfrak{S}_m^\beta$  are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^* \mathbf{A}|^{\beta(m-1)/2+1} (d\mathbf{X}). \tag{17}$$

**Proposition 2.3** (Singular value decomposition, SVD). *Let  $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$  be matrix of functionally independent variables, such that  $\mathbf{X} = \mathbf{W}^* \mathbf{D} \mathbf{V}_1$  with  $\mathbf{V}_1 \in \mathcal{V}_{m,n}^\beta$ ,  $\mathbf{W} \in \mathfrak{U}^\beta(m)$  and  $\mathbf{D} = \text{diag}(d_1, \dots, d_m) \in \mathfrak{D}_m^1$ ,  $d_1 > \dots > d_m > 0$ . Then*

$$(d\mathbf{X}) = 2^{-m} \pi^\tau \prod_{i=1}^m d_i^{\beta(n-m+1)-1} \prod_{i < j}^m (d_i^2 - d_j^2)^\beta (d\mathbf{D}) (\mathbf{V}_1 d\mathbf{V}_1^*) (\mathbf{W} d\mathbf{W}^*), \tag{18}$$

where

$$\tau = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

**Proposition 2.4.** Let  $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$  be matrix of functionally independent variables, and  $\mathbf{S} = \mathbf{X}\mathbf{X}^* \in \mathfrak{P}_m^\beta$ . Then

$$(d\mathbf{X}) = 2^{-m} |\mathbf{S}|^{\beta(n-m+1)/2-1} (d\mathbf{S})(\mathbf{V}_1 d\mathbf{V}_1^*), \quad (19)$$

with  $\mathbf{V}_1 \in \mathcal{V}_{m,n}^\beta$ .

### 3 Matricvariate $T$ -Riesz distribution

In this section two versions of the matricvariate  $T$ -Riesz distribution and the corresponding generalised beta type II distributions are obtained.

A detailed discussion of Riesz distribution may be found in Hassairi and Lajmi (2001) and Díaz-García (2013a). In addition the Kotz-Riesz distribution is studied in detail in Díaz-García (2013c). For your convenience, we adhere to standard notation stated in Díaz-García (2013a,c).

**Theorem 3.1.** Let  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers, and  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers. Also define  $\mathbf{T} \in \mathcal{L}_{m,n}^\beta$  as

$$\mathbf{T} = \mathbf{L}^{-1} \mathbf{Y} + \boldsymbol{\mu},$$

where  $\mathbf{L}$  is any square root of  $\mathbf{V}$  such that  $\mathbf{L}^* \mathbf{L} = \mathbf{V} \sim \mathcal{R}_m^{\beta,I}(\nu\beta/2, \kappa, \boldsymbol{\Xi})$ ,  $\boldsymbol{\Xi} \in \mathfrak{P}_m^\beta$  and  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 - k_m$ ; independent of  $\mathbf{Y} \sim \mathcal{KR}_{m \times n}^{\beta,I}(\tau, \mathbf{0}, \mathbf{I}_m, \boldsymbol{\Sigma})$ ,  $\boldsymbol{\Sigma} \in \mathfrak{P}_n^\beta$  and  $\text{Re}(n\beta/2) > (m-1)\beta/2 - t_m$ . Then the density of  $\mathbf{T}$  is

$$\begin{aligned} & \propto |\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{T} - \boldsymbol{\mu})^*|^{-(n+\nu)\beta/2} q_\tau[(\mathbf{T} - \boldsymbol{\mu})(\mathbf{T} - \boldsymbol{\mu})^*] \\ & \times q_{\kappa+\tau}^{-1} [\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{T} - \boldsymbol{\mu})^*] (d\mathbf{T}), \end{aligned} \quad (20)$$

where the constant of proportionality is

$$\frac{\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2} \mathcal{B}_m^\beta[\nu\beta/2, \kappa; n\beta/2, \tau] |\boldsymbol{\Xi}|^{\nu\beta/2} |\boldsymbol{\Sigma}|^{m\beta/2} q_\kappa(\boldsymbol{\Xi}) q_\tau(\boldsymbol{\Sigma})},$$

which is shall be termed the matricvariate  $T$ -Riesz distribution type I and is denoted as

$$\mathbf{T} \sim \mathcal{TR}_{m \times n}^{\beta,I}(\nu, \kappa, \tau, \boldsymbol{\mu}, \boldsymbol{\Xi}, \boldsymbol{\Sigma}).$$

*Proof.* From Díaz-García (2013a,c), the joint density of  $\mathbf{V}$  and  $\mathbf{Y}$  is

$$\begin{aligned} & \propto |\mathbf{V}|^{(\nu-m+1)\beta/2-1} \text{etr}\{-\beta(\boldsymbol{\Xi}^{-1}\mathbf{V} + \boldsymbol{\Sigma}^{-1}\mathbf{Y}^*\mathbf{Y})\} \\ & \times q_\kappa(\mathbf{V}) q_\tau(\boldsymbol{\Sigma}^{-1/2}\mathbf{Y}^*\mathbf{Y}\boldsymbol{\Sigma}^{-1/2}) (d\mathbf{V})(d\mathbf{Y}), \end{aligned}$$

where the constant of proportionality given by

$$c = \frac{\beta^{\nu m\beta/2 + \sum_{i=1}^m k_i}}{\Gamma_m^\beta[\nu\beta/2, \kappa] |\boldsymbol{\Xi}|^{\beta\nu/2} q_\kappa(\boldsymbol{\Xi})} \cdot \frac{\beta^{mn\beta/2 + \sum_{i=1}^m t_i} \Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2} \Gamma_m^\beta[n\beta/2, \tau] |\boldsymbol{\Sigma}|^{\beta m/2}}.$$

Making the change of variable  $\mathbf{Y} = \mathbf{L}(\mathbf{T} - \boldsymbol{\mu})$ , where  $\mathbf{V} = \mathbf{L}^* \mathbf{L}$ , then by (16)

$$(d\mathbf{V})(d\mathbf{Y}) = |\mathbf{L}\mathbf{L}^*|^{n\beta/2} (d\mathbf{V})(d\mathbf{T}) = |\mathbf{V}|^{n\beta/2} (d\mathbf{V})(d\mathbf{T}).$$



Thus, the joint density of  $\mathbf{V}$  and  $\mathbf{T}$  is

$$\begin{aligned} & \propto |\mathbf{V}|^{\beta(\nu+n-m+1)/2-1} \text{etr} \left\{ -\beta \left[ \boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1} (\mathbf{T} - \boldsymbol{\mu})^* \right] \mathbf{V} \right\} \cdot \\ & \times q_{\kappa}(\mathbf{V}) q_{\tau} \left( \boldsymbol{\Sigma}^{-1/2} (\mathbf{T} - \boldsymbol{\mu}) (\mathbf{T} - \boldsymbol{\mu})^* \boldsymbol{\Sigma}^{-1/2} \right) (d\mathbf{V})(d\mathbf{T}). \end{aligned}$$

Finally, integrating over  $\mathbf{V} \in \mathfrak{P}_m^{\beta}$  using the density of Riesz distribution (Díaz-García, 2013a), we have

$$\begin{aligned} & \frac{\Gamma_m^{\beta}[(n+\nu)\beta/2, \kappa + \tau]}{\beta^{(\nu+n)m\beta/2 + \sum_{i=1}^m (k_i + t_i)}} |\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1} (\mathbf{T} - \boldsymbol{\mu})^*|^{-(n+\nu)\beta/2} \\ & \times q_{\tau}(\boldsymbol{\Sigma}^{-1/2} (\mathbf{T} - \boldsymbol{\mu}) (\mathbf{T} - \boldsymbol{\mu})^* \boldsymbol{\Sigma}^{-1/2}) q_{\kappa+\tau} \left[ (\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1} (\mathbf{T} - \boldsymbol{\mu})^*)^{-1} \right] (d\mathbf{T}), \end{aligned}$$

from which the desired result is obtained applying (7) and observing that by (7) and (12),

$$q_{\tau}(\boldsymbol{\Sigma}^{-1/2} (\mathbf{T} - \boldsymbol{\mu}) (\mathbf{T} - \boldsymbol{\mu})^* \boldsymbol{\Sigma}^{-1/2}) = \frac{q_{\tau}((\mathbf{T} - \boldsymbol{\mu}) (\mathbf{T} - \boldsymbol{\mu})^*)}{q_{\tau}(\boldsymbol{\Sigma})}.$$

□

□

**Theorem 3.2.** Let  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers, and  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers. Also define  $\mathbf{T} \in \mathcal{L}_{m,n}^{\beta}$  as

$$\mathbf{T} = \mathbf{L}^{-1} \mathbf{Y} + \boldsymbol{\mu},$$

where  $\mathbf{L}$  is any square root of  $\mathbf{V}$  such that  $\mathbf{L}^* \mathbf{L} = \mathbf{V} \sim \mathcal{R}_m^{\beta, II}(\nu\beta/2, \kappa, \boldsymbol{\Xi})$ ,  $\boldsymbol{\Xi} \in \mathfrak{P}_m^{\beta}$  and  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 + k_1$ ; independent of  $\mathbf{Y} \sim \mathcal{KR}_{m \times n}^{\beta, II}(\tau, \mathbf{0}, \mathbf{I}_m, \boldsymbol{\Sigma})$ ,  $\boldsymbol{\Sigma} \in \mathfrak{P}_n^{\beta}$  and  $\text{Re}(n\beta/2) > (m-1)\beta/2 + t_1$ . Then the density of  $\mathbf{T}$  is

$$\begin{aligned} & \propto |\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1} (\mathbf{T} - \boldsymbol{\mu})^*|^{-(n+\nu)\beta/2} q_{\tau}^{-1} [(\mathbf{T} - \boldsymbol{\mu}) (\mathbf{T} - \boldsymbol{\mu})^*] \\ & \times q_{\kappa+\tau} [\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1} (\mathbf{T} - \boldsymbol{\mu})^*] (d\mathbf{T}), \end{aligned} \quad (21)$$

where the constant of proportionality is given by

$$c_2 = \frac{\Gamma_m^{\beta}[n\beta/2] q_{\kappa}(\boldsymbol{\Xi}) q_{\tau}(\boldsymbol{\Sigma})}{\pi^{mn\beta/2} \mathcal{B}_m^{\beta}[\nu\beta/2, -\kappa; n\beta/2, -\tau] |\boldsymbol{\Xi}|^{\nu\beta/2} |\boldsymbol{\Sigma}|^{m\beta/2}},$$

which shall be termed the matrixvariate  $T$ -Riesz distribution type II and is denoted as

$$\mathbf{T} \sim \mathcal{TR}_{m \times n}^{\beta, II}(\nu, \kappa, \tau, \boldsymbol{\mu}, \boldsymbol{\Xi}, \boldsymbol{\Sigma}).$$

*Proof.* The proof is similar to that given for Theorem 3.1. □

Now, observe that:

1. analogously to Dickey (1967, eq. (2.5))

$$\frac{\Gamma_m^{\beta}[n\beta/2]}{\pi^{mn\beta/2} \mathcal{B}_m^{\beta}[n\beta/2, \tau; \nu\beta/2, \kappa]} = \frac{\Gamma_n^{\beta}[m\beta/2]}{\pi^{mn\beta/2} \mathcal{B}_n^{\beta}[m\beta/2, \tau; \beta(n+\nu-m)/2, \kappa]},$$

- 2.

$$|\boldsymbol{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\nu}) \boldsymbol{\Sigma}^{-1} (\mathbf{T} - \boldsymbol{\nu})^*| = |\boldsymbol{\Xi}|^{-1} |\boldsymbol{\Sigma}|^{-1} |\boldsymbol{\Sigma} + (\mathbf{T} - \boldsymbol{\nu})^* \boldsymbol{\Xi} (\mathbf{T} - \boldsymbol{\nu})|,$$

3. Considering the next extension defining  $q_\kappa(\mathbf{A})$  as: if  $\lambda_1, \dots, \lambda_r$ , are the non null eigenvalues of  $\mathbf{A} \in \mathfrak{S}_m^\beta$ , then

$$q_\kappa(\mathbf{A}) = \prod_{i=1}^r \lambda_i^{k_i}.$$

then,  $q_{\kappa+\tau}^{-1} [\mathbf{\Xi}^{-1} + (\mathbf{T} - \boldsymbol{\mu})\mathbf{\Sigma}^{-1}(\mathbf{T} - \boldsymbol{\mu})^*]$  can be written as

$$= q_{\kappa+\tau}(\mathbf{\Xi})q_{\kappa+\tau}(\mathbf{\Sigma})q_{\kappa+\tau}^{-1} [\mathbf{\Sigma} + (\mathbf{T} - \boldsymbol{\mu})^*\mathbf{\Xi}(\mathbf{T} - \boldsymbol{\mu})],$$

and applying (9), the density (20) can be expressed alternatively as

$$\begin{aligned} & \frac{\Gamma_n^\beta[m\beta/2]|\mathbf{\Xi}|^{n\beta/2}|\mathbf{\Sigma}|^{(\nu+n-m)\beta/2}q_\kappa(\mathbf{\Xi})q_\tau(\mathbf{\Sigma})}{\pi^{mn\beta/2}\mathcal{B}_n^\beta[m\beta/2, \tau; (\nu+n-m)\beta/2, \kappa]} |\mathbf{\Sigma} + (\mathbf{T} - \boldsymbol{\mu})^*\mathbf{\Xi}(\mathbf{T} - \boldsymbol{\mu})|^{-(n+\nu)\beta/2} \\ & \times q_\tau[(\mathbf{T} - \boldsymbol{\mu})(\mathbf{T} - \boldsymbol{\mu})^*] q_{\kappa+\tau}^{-1}[\mathbf{\Sigma} + (\mathbf{T} - \boldsymbol{\mu})^*\mathbf{\Xi}(\mathbf{T} - \boldsymbol{\mu})] (d\mathbf{T}). \end{aligned} \quad (22)$$

**Corollary 3.1.** Let  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers, and  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers. Also define  $\mathbf{T} \in \mathcal{L}_{m,n}^\beta$  as

$$\mathbf{T} = \mathbf{X}\mathbf{L}_1^{-1} + \boldsymbol{\mu}$$

where  $\mathbf{L}_1$  is any square root of  $\mathbf{U}$  such that  $\mathbf{L}_1\mathbf{L}_1^* = \mathbf{U} \sim \mathcal{R}_n^{\beta,I}((\nu+n-m)\beta/2, \kappa, \mathbf{\Sigma}^{-1})$ ,  $\mathbf{\Sigma} \in \mathfrak{P}_n^\beta$  and  $\text{Re}((\nu+n-m)\beta/2) > (m-1)\beta/2 - k_m$ ; independent of  $\mathbf{X} \sim \mathcal{K}_{m \times n}^{\beta,I}(\tau, \mathbf{0}, \mathbf{\Xi}^{-1}, \mathbf{I}_n)$ ,  $\mathbf{\Xi} \in \mathfrak{P}_m^\beta$  and  $\text{Re}(n\beta/2) > (m-1)\beta/2 - t_m$ . Then the density of  $\mathbf{T}$  is given by (22), this is

$$\mathbf{T} \sim \mathcal{TR}_{m \times n}^{\beta,I}(\nu, \kappa, \tau, \boldsymbol{\mu}, \mathbf{\Xi}, \mathbf{\Sigma}).$$

*Proof.* The proof is analogous to the proof of Theorem 3.1. □ □

Similar to Corollary 1, now assuming distributions Kotz-Riesz and Riesz type II the following result is obtained.

**Corollary 3.2.** Let  $\kappa = (k_1, k_2, \dots, k_m)$ ,  $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$ ,  $k_1, k_2, \dots, k_m$  are nonnegative integers, and  $\tau = (t_1, t_2, \dots, t_m)$ ,  $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$ ,  $t_1, t_2, \dots, t_m$  are nonnegative integers. Also define  $\mathbf{T} \in \mathcal{L}_{m,n}^\beta$  as

$$\mathbf{T} = \mathbf{X}\mathbf{L}_1^{-1} + \boldsymbol{\mu}$$

where  $\mathbf{L}_1$  is any square root of  $\mathbf{U}$  such that  $\mathbf{L}_1\mathbf{L}_1^* = \mathbf{U} \sim \mathcal{R}_n^{\beta,II}((\nu+n-m)\beta/2, \kappa, \mathbf{\Sigma}^{-1})$ ,  $\mathbf{\Sigma} \in \mathfrak{P}_n^\beta$  and  $\text{Re}((\nu+n-m)\beta/2) > (m-1)\beta/2 + k_1$ ; independent of  $\mathbf{X} \sim \mathcal{K}_{m \times n}^{\beta,II}(\tau, \mathbf{0}, \mathbf{\Xi}^{-1}, \mathbf{I}_n)$ ,  $\mathbf{\Xi} \in \mathfrak{P}_m^\beta$  and  $\text{Re}(n\beta/2) > (m-1)\beta/2 + t_1$ . Then the density of  $\mathbf{T}$  is given by

$$\begin{aligned} & \frac{\Gamma_m^\beta[n\beta/2]|\mathbf{\Xi}|^{n\beta/2}|\mathbf{\Sigma}|^{(\nu+n-m)\beta/2}}{\pi^{mn\beta/2}\mathcal{B}_n^\beta[n\beta/2, -\tau; (\nu+n-m)\beta/2, -\kappa]q_\kappa(\mathbf{\Xi})q_\tau(\mathbf{\Sigma})} \\ & \times |\mathbf{\Sigma} + (\mathbf{T} - \boldsymbol{\mu})^*\mathbf{\Xi}(\mathbf{T} - \boldsymbol{\mu})|^{-(n+\nu)\beta/2} q_\tau^{-1}[(\mathbf{T} - \boldsymbol{\mu})(\mathbf{T} - \boldsymbol{\mu})^*] \\ & \times q_{\kappa+\tau}[\mathbf{\Sigma} + (\mathbf{T} - \boldsymbol{\mu})^*\mathbf{\Xi}(\mathbf{T} - \boldsymbol{\mu})] (d\mathbf{T}). \end{aligned} \quad (23)$$

this is

$$\mathbf{T} \sim \mathcal{TR}_{m \times n}^{\beta,II}(\nu, \kappa, \tau, \boldsymbol{\mu}, \mathbf{\Xi}, \mathbf{\Sigma}).$$

Observe that if theorems 3.1 and 3.2 and corollaries 3.1 and 3.2 are defined  $\kappa = (0, \dots, 0)$  and  $\tau = (0, \dots, 0)$ , results in Díaz-García and Gutiérrez-Jáimez (2012) and Dickey (1967) (with  $\beta = 1$ ) are obtained as particular cases.

Let  $\mathbf{F} \in \mathfrak{P}_m^\beta$  be defined as  $\mathbf{F} = \mathbf{T}\mathbf{T}^*$ , such that  $\mathbf{T} \sim \mathcal{TR}_{m \times n}^{\beta, I}(\nu, \kappa, \tau, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$  with  $n \geq m$ . Then, under the conditions of Theorem 3.1 and Corollary 3.1, we have

$$\begin{aligned}\mathbf{F} &= \mathbf{L}^{-1}\mathbf{Y}\mathbf{Y}^*(\mathbf{L}^{-1})^* = \mathbf{L}^{-1}\mathbf{S}(\mathbf{L}^{-1})^* \\ &= \mathbf{X}\mathbf{U}^{-1}\mathbf{X}^*,\end{aligned}$$

where  $\mathbf{S} = \mathbf{Y}\mathbf{Y}^* \sim \mathcal{R}_m^{\beta, I}(n\beta/2, \tau, \mathbf{I}_m)$ , with  $\text{Re}(n\beta/2) > (m-1)\beta/2 - t_m$ . Thus:

**Theorem 3.3.** *The density of  $\mathbf{F}$  is*

$$\propto |\mathbf{F}|^{(n-m+1)\beta/2-1} |\mathbf{I}_m + \mathbf{F}|^{-(n+\nu)\beta/2} q_{\kappa+\tau}^{-1}(\mathbf{I}_m + \mathbf{F}) q_\tau(\mathbf{F}) (d\mathbf{F}), \quad (24)$$

with the constant of proportionality defined by

$$\frac{1}{\mathcal{B}_m^\beta[\nu\beta/2, \kappa; n\beta/2, \tau]},$$

where  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 - k_m$  and  $\text{Re}(n\beta/2) > (m-1)\beta/2 - t_m$ . It is said that  $\mathbf{F}$  has a matricvariate c-beta-Riesz type II distribution.

*Proof.* The proof follows from (20) by applying (19) and then (2).  $\square$   $\square$

Analogously, under the conditions of Theorem 3.2 and Corollary 3.2, we have the following result.

**Theorem 3.4.** *The density of  $\mathbf{F}$  is*

$$\propto |\mathbf{F}|^{(n-m+1)\beta/2-1} |\mathbf{I}_m + \mathbf{F}|^{-(n+\nu)\beta/2} q_{\kappa+\tau}(\mathbf{I}_m + \mathbf{F}) q_\tau^{-1}(\mathbf{F}) (d\mathbf{F}), \quad (25)$$

where the constant of proportionality is

$$\frac{1}{\mathcal{B}_m^\beta[\nu\beta/2, -\kappa; n\beta/2, -\tau]},$$

where  $\text{Re}(\nu\beta/2) > (m-1)\beta/2 + k_1$  and  $\text{Re}(n\beta/2) > (m-1)\beta/2 + t_1$ . It is said that  $\mathbf{F}$  has a matricvariate k-beta-Riesz type II distribution.

Observe that, results in theorems 3.3 and 3.4 were obtained by Díaz-García (2013b) via an alternative way.

Now suppose that  $n < m$  and defined  $\tilde{\mathbf{F}} = \mathbf{T}^*\mathbf{T} \in \mathfrak{P}_n^\beta$  then, under the conditions of Theorem 3.1 and Corollary 3.1 we have

$$\begin{aligned}\tilde{\mathbf{F}} &= \mathbf{X}^*\mathbf{U}^{-1}\mathbf{X} \\ &= \mathbf{L}_1^{-1}\mathbf{X}^*\mathbf{X}(\mathbf{L}_1^{-1})^* = \mathbf{L}_1^{-1}\mathbf{S}_1(\mathbf{L}_1^{-1})^*\end{aligned}$$

with  $\mathbf{S}_1 = \mathbf{X}^*\mathbf{X} \sim \mathcal{R}_n^{\beta, I}(m\beta/2, \tau, \mathbf{I}_n)$ ,  $\text{Re}(m\beta/2) > (n-1)\beta/2 - t_n$ . Thus:

**Theorem 3.5.**  *$\tilde{\mathbf{F}}$  has density*

$$\propto |\tilde{\mathbf{F}}|^{(m-n+1)\beta/2-1} |\mathbf{I}_n + \tilde{\mathbf{F}}|^{-(n+\nu)\beta/2} q_{\kappa+\tau}^{-1}(\mathbf{I}_n + \tilde{\mathbf{F}}) q_\tau(\tilde{\mathbf{F}}) (d\tilde{\mathbf{F}}), \quad (26)$$

with constant of proportionality

$$\frac{1}{\mathcal{B}_m^\beta[(\nu+n-m)\beta/2, \kappa; m\beta/2, \tau]},$$

where  $\text{Re}[(\nu+n-m)\beta/2] > (n-1)\beta/2 - k_n$  and  $\text{Re}(m\beta/2) > (n-1)\beta/2 - t_n$ . And we say that  $\tilde{\mathbf{F}}$  has a matricvariate c-beta-Riesz type II distribution.

Similarly, under the conditions of Theorem 3.2 and Corollary 3.2, we obtain:

**Theorem 3.6.**  $\tilde{\mathbf{F}}$  has density

$$\propto |\tilde{\mathbf{F}}|^{(m-n+1)\beta/2-1} |\mathbf{I}_n + \tilde{\mathbf{F}}|^{-(n+\nu)\beta/2} q_{\kappa+\tau}(\mathbf{I}_n + \tilde{\mathbf{F}}) q_{\tau}^{-1}(\tilde{\mathbf{F}})(d\tilde{\mathbf{F}}), \quad (27)$$

with constant of proportionality

$$\frac{1}{\mathcal{B}_m^{\beta}[(\nu+n-m)\beta/2, -\kappa; m\beta/2, -\tau]},$$

where  $\text{Re}[(\nu+n-m)\beta/2] > (n-1)\beta/2 + k_1$  and  $\text{Re}(m\beta/2) > (n-1)\beta/2 + t_1$ . Furthermore, we say that  $\tilde{\mathbf{F}}$  has a matricvariate k-beta-Riesz type II distribution.

Proofs of theorem 3.5 and 3.6 are similar as those given for Theorem 3.3. Or observe that densities (26) and (27) can be obtained from densities (24) and (25) respectively, by the following substitutions, see Muirhead (1982, Eq. (7), p. 455) and Srivastava & Khatri (1979, p. 96),

$$m \rightarrow n, \quad n \rightarrow m, \quad \nu \rightarrow \nu + n - m. \quad (28)$$

To conclude this section, suppose that  $\mathbf{M} \in \mathcal{L}_{m \times m}^{\beta}$  is any square root of the constant matrix  $\mathbf{\Delta} = \mathbf{M}^* \mathbf{M} \in \mathfrak{P}_m^{\beta}$ . Also, define  $\mathbf{Z} = \mathbf{M}^* \mathbf{F} \mathbf{M}$ , therefore:

**Corollary 3.3.** 1. If  $\mathbf{F}$  has a c-beta-Riesz type II distribution, the density of  $\mathbf{Z}$  is

$$\propto |\mathbf{Z}|^{(n-m+1)\beta/2-1} |\mathbf{\Delta} + \mathbf{Z}|^{-(n+\nu)\beta/2} q_{\kappa+\tau}^{-1}(\mathbf{\Delta} + \mathbf{Z}) q_{\tau}(\mathbf{Z})(d\mathbf{Z}), \quad (29)$$

with constant of proportionality

$$\frac{|\mathbf{\Delta}|^{\nu\beta/2} q_{\kappa}(\mathbf{\Delta})}{\mathcal{B}_m^{\beta}[\nu\beta/2, \kappa; n\beta/2, \tau]},$$

where  $\text{Re}[\nu\beta/2] > (m-1)\beta/2 - k_m$  and  $\text{Re}(m\beta/2) > (m-1)\beta/2 - t_m$ .  $\mathbf{Z}$  is said to have a nonstandardised matricvariate c-beta-Riesz type II distribution.

2. If  $\mathbf{F}$  has a k-beta-Riesz type II distribution, the density of  $\mathbf{Z}$  is

$$\propto |\mathbf{Z}|^{(n-m+1)\beta/2-1} |\mathbf{\Delta} + \mathbf{Z}|^{-(n+\nu)\beta/2} q_{\kappa+\tau}(\mathbf{\Delta} + \mathbf{Z}) q_{\tau}^{-1}(\mathbf{Z})(d\mathbf{Z}), \quad (30)$$

with constant of proportionality

$$\frac{|\mathbf{\Delta}|^{\nu\beta/2}}{\mathcal{B}_m^{\beta}[\nu\beta/2, -\kappa; n\beta/2, -\tau] q_{\kappa}(\mathbf{\Delta})},$$

where  $\text{Re}[\nu\beta/2] > (m-1)\beta/2 + k_1$  and  $\text{Re}(m\beta/2) > (m-1)\beta/2 + t_1$ .  $\mathbf{Z}$  is said to have a non standardised matricvariate k-beta-Riesz type II distribution.

*Proof.* Proof follows from (24) and (25), respectively, by applying (17).  $\square$   $\square$

If in theorems and corollaries in this section are defined  $k = 0$  and  $\tau = (0, \dots, 0)$ , results in Díaz-García and Gutiérrez-Jáimez (2012) are obtained as particular cases.

## 4 Singular value densities

In this section, the joint densities of the singular values of matrices  $\mathbf{T}$  and  $\tilde{\mathbf{T}}$  types I and II are derived. In addition, and as a direct consequence, the joint densities of the eigenvalues of  $\mathbf{F}$  and  $\tilde{\mathbf{F}}$  types I and II are obtained for real normed division algebras.

**Theorem 4.1.** 1. Let  $\mathbf{T} \sim \mathcal{TR}_{m \times n}^{\beta, I}(\nu, \kappa, \tau, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ , and let  $\delta_1, \dots, \delta_m$  be its singular values,  $\delta_1 > \dots > \delta_m > 0$ . Then its joint density is

$$\propto \prod_{i=1}^m \delta_i^{(n-m+1)\beta+2k_i-1} (1 + \delta_i^2)^{-\beta(\nu+n)/2-k_i-t_i} \prod_{i < j}^m (\delta_i^2 - \delta_j^2)^\beta \quad (31)$$

with constant of proportionality given by

$$\frac{2^m \pi^{m^2/2+\tau}}{\Gamma_m^\beta[\beta m/2] \mathcal{B}_m^\beta[\nu\beta/2, \kappa; n\beta/2, \tau]}.$$

2. Let  $\mathbf{T} \sim \mathcal{TR}_{m \times n}^{\beta, II}(\nu, \kappa, \tau, \mathbf{0}, \mathbf{I}_m, \mathbf{I}_n)$ , and let  $\delta_1, \dots, \delta_m$  be its singular values,  $\delta_1 > \dots > \delta_m > 0$ . Then its joint density is

$$\propto \prod_{i=1}^m \delta_i^{(n-m+1)\beta-2k_i-1} (1 + \delta_i^2)^{-\beta(\nu+n)/2+k_i+t_i} \prod_{i < j}^m (\delta_i^2 - \delta_j^2)^\beta \quad (32)$$

with constant of proportionality given by

$$\frac{2^m \pi^{m^2/2+\tau}}{\Gamma_m^\beta[\beta m/2] \mathcal{B}_m^\beta[\nu\beta/2, -\kappa; n\beta/2, -\tau]}.$$

Where  $\tau$  is defined in Lemma 2.3.

*Proof.* This follows immediately from (20), first using (18) and then applying (2).  $\square \quad \square$

Joint densities of the singular values of  $\tilde{\mathbf{T}}$  types I and II are obtained from (31) and (32) respectively, after making the substitutions (28).

Finally, observe that  $\delta_i = \sqrt{\text{eig}_i(\mathbf{T}\mathbf{T}^*)}$  and  $\alpha_i = \sqrt{\text{eig}_i(\mathbf{T}_1\mathbf{T}_1^*)}$ , where  $\text{eig}_i(\mathbf{A})$ ,  $i = 1, \dots, m$ , denotes the  $i$ -th eigenvalue of  $\mathbf{A}$ . Let  $\lambda_i = \text{eig}_i(\mathbf{T}\mathbf{T}^*) = \text{eig}_i(\mathbf{F})$ , observing that, for example,  $\delta_i = \sqrt{\lambda_i}$ . Then

$$\bigwedge_{i=1}^m d\delta_i = \bigwedge_{i=1}^m 2^{-m} \prod_{i=1}^m \lambda_i^{-1/2} d\lambda_i,$$

the corresponding joint densities of  $\lambda_1, \dots, \lambda_m$ ,  $\lambda_1 > \dots > \lambda_m > 0$  types I and II are obtained from (31) and (32) respectively as

$$1. \quad \propto \prod_{i=1}^m \lambda_i^{(n-m+1)\beta/2+k_i-1} (1 + \lambda_i)^{-\beta(\nu+n)/2-k_i-t_i} \prod_{i < j}^m (\lambda_i - \lambda_j)^\beta \quad (33)$$

with constant of proportionality given by

$$\frac{\pi^{m^2/2+\tau}}{\Gamma_m^\beta[\beta m/2] \mathcal{B}_m^\beta[\nu\beta/2, \kappa; n\beta/2, \tau]}.$$

2.

$$\propto \prod_{i=1}^m \lambda_i^{(n-m+1)\beta/2-k_i-1} (1+\lambda_i)^{-\beta(\nu+n)/2+k_i+t_i} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \quad (34)$$

with constant of proportionality given by

$$\frac{\pi^{m^2/2+\tau}}{\Gamma_m^\beta[\beta m/2] \mathcal{B}_m^\beta[\nu\beta/2, -\kappa; n\beta/2, -\tau]}.$$

Densities (33) and (34) were obtained by alternative way in Díaz-García (2013b).

## Conclusions

Note that if in sections 3 and 4 is defined  $\tau = (p, \dots, p)$  the corresponding results for the matricvariate Kotz type distribution are obtained as a particular case, see Fang and Li (1999).

In addition, note that the real dimension of real normed division algebras can be expressed as powers of 2,  $\beta = 2^n$  for  $n = 0, 1, 2, 3$ . On the other hand, as it can be reviewed in Kabe (1984), the results obtained in this work can be extended to the hypercomplex cases; that is, for complex, bicomplex, biquaternion and bioctonion (or sedenionic) algebras, which of course are not division algebras (except the complex algebra). Note also, that hypercomplex algebras are obtained by replacing the real numbers with complex numbers in the construction of real normed division algebras. Thus, the results for hypercomplex algebras are obtained by simply replacing  $\beta$  with  $2\beta$  in our results. Alternatively, following Kabe (1984), we can conclude that, our results are true for ‘ $2^n$ -ions’,  $n = 0, 1, 2, 3, 4, 5$ , emphasising that only for  $n = 0, 1, 2, 3$  the corresponding algebras are real normed division algebras.

However, although these generalisations may have a theoretical interest, from the practical point of view, we must keep in mind the words of Baez (2002), *there is still no proof that the octonions are useful for understanding the real world*. We can only hope that eventually this question shall be answered one way or another. Also, for the sake of completeness, in the present article the case of octonions is considered, but the veracity of the results obtained for this case can only be conjectured; since there are still many problems under study in the context of the octonions.

## Acknowledgements

This research work was partially supported by IDI-Spain, Grants No. MTM2011-28962. This paper was written during J. A. Díaz-García’s stay as a visiting professor at the Department of Statistics and O. R. of the University of Granada, Spain.

## References

- Baez J C (2002) The octonions. Bull. Amer. Math. Soc. 39: 145–205.
- Díaz-García J A (2013a) Riesz distributions. Metrika DOI 10.1007/s00184-013-0449-5.
- Díaz-García J A (2013b) Distributions on symmetric cones II: Beta-Riesz distributions. Cornell University Library, <http://arxiv.org/abs/1301.4525>.

- Díaz-García J A (2013c) A generalised Kotz type distribution and Riesz distribution. Cornell University Library, <http://arxiv.org/abs/1304.5292>.
- Díaz-García J A, Gutiérrez-Jáimez R (2011) On Wishart distribution: Some extensions. *Linear Algebra Appl.* 435: 1296-1310.
- Díaz-García J A, Gutiérrez-Jáimez R (2012) Matricvariate and matrix multivariate T distributions and associated distributions. *Metrika*, 75(7): 963-976.
- Dickey J M (1967) Matricvariate generalizations of the multivariate  $t$ -distribution and the inverted multivariate  $t$ -distribution. *Ann. Math.Statist.* 38: 511-518.
- Edelman A, Rao R R (2005) Random matrix theory. *Acta Numerica* 14, 233-297.
- Fang K T, Li R (1999) Bayesian statistical inference on elliptical matrix distributions. *J. Multivariate Anal.* 70: 66-85.
- Fang K T, Zhang Y T (1990) *Generalized Multivariate Analysis*. Science Press, Beijing, Springer-Verlang.
- Faraut J, Korányi A (1994) *Analysis on symmetric cones*. Oxford Mathematical Monographs, Clarendon Press, Oxford.
- Gross K I, and Richards D St P (1987) Special functions of matrix argument I: Algebraic induction zonal polynomials and hypergeometric functions. *Trans. Amer. Math. Soc.* 301(2): 475-501.
- Gupta A K, and Varga T (1993) *Elliptically Contoured Models in Statistics*. Kluwer Academic Publishers, Dordrecht.
- Gupta A K, Nagar D K (2000) *Matrix variate distributions*. Chapman & Hall/CR, New York.
- Hassairi A, Lajmi S (2001) Riesz exponential families on symmetric cones. *J. Theoret. Probab.* 14: 927-948.
- Kabe D G, Classical statistical analysis based on a certain hypercomplex multivariate normal distribution. *Metrika* 31: 63-76.
- Khatri C G (1966) On certain distribution problems based on positive definite quadratic functions in normal vector, *Ann. Math. Statist. A* 37: 468-479.
- Muirhead R J (1982) *Aspects of Multivariate Statistical Theory*, John Wiley & Sons, New York.
- Neukirch J, Prestel A, Remmert R (1990) *Numbers*, GTM/RIM 123, H.L.S. Orde, tr. NWU-user.
- Press S J (1982) *Applied Multivariate Analysis: Using Bayesian and Frequentist Methods of Inference*. Second Edition, Robert E. Krieger Publishing Company, Malabar, Florida.
- Srivastava S M, Khatri C G (1979) *An introduction to multivariate statistics*. North Holland, New York, 1979.